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NOTE

A Short Proof of a Theorem Concerning Degree Sums and Connectivity on Hamiltonian Graphs*

Bing Wei

Institute of Systems Science, Academia Sinica, Beijing, 100080, People's Republic of China

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D. Bauer, H. J. Broersma, R. Li, and H. J. Veldman proved that if G is a 2-con-

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ or just by V; the set of edges by $E(G)$ or just by E. We use $|G|$ as a symbol for the cardinality of $V(G)$. If H and S are subsets of $V(G)$ or subgraph of G, we denote by $N_H(S)$ the set of vertices in H which are adjacent to some vertex in S, and set $d_H(S)=|N_H(S)|$. In particular, when $H=G$, $S=\{u\}$, then let $N_G(u)=N(u)$ and set $d_G(u)=d(u)$. Let κ denote the vertex connectivity of G. We call a cycle C *dominating*, if $V(G)-V(C)$ is an independent set of G. As usual, we use δ for the minimum degree of vertices of a graph. Now set $\sigma_3=\min\{\sum_{i=1}^{3} d(x_i)\colon \{x_1, x_2, x_3\}$ is an independent set of $G\}$.

In 1981, Häggkvist and Nicoghossian proved:

THEOREM 1 [3]. *Let G be a 2-connected graph on n vertices such that $\delta \geqslant (n+\kappa)/3$. Then G is hamiltonian.*

Theorem 1 was generalized by Bauer, Broersma, Li Rao, and Veldman as follows.

THEOREM 2 [1]. *Let G be a 2-connected graph on n vertices such that $\sigma_3 \geqslant n+\kappa$. Then G is hamiltonian.*

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In this note, we give a short proof of Theorem 2. The method we used in the proof may be applied in finding or proving other results which concern degree sums and the connectivity of graphs. In order to prove Theorem 2, we need the following.

THEOREM 3 [2]. *Let G be a 2-connected graph on n vertices such that $\sigma_3 \geqslant n+2$. Then every longest cycle of G is dominating.*

For the rest of the proof, suppose that G is a nonhamiltonian graph satisfying the hypothesis of Theorem 2. Let C be a longest cycle of G. We assign an orientation to C and, for any $v \in V(C)$, denote by v^- the vertex preceding v on C and v^+ the vertex following v; this notation is extended to sets. For u, v on C, let $C[u, v]$ denote the subpath of C from u to v (in the chosen direction). For $C[u^+, v]$ we also write $C(u, v]$, similarly, $C[u, v) = C[u, v^-]$.

Let $v \in V(G-C)$. Since $\kappa \geqslant 2$, by Theorem 3, $N(v) \subseteq V(C)$. Let $X = N(v) = \{x_1, x_2, ..., x_t\}$ in order around C and $T = X^- = \{x_1^-, ..., x_t^-\}$. Then $|T| = d(v) \geqslant \kappa \geqslant 2$. Set $C_i = C[x_i, x_{i+1})$ for $i < t$ and $C_t = C[x_t, x_1)$. The maximality of C implies the following well-known properties:

LEMMA 4. (i) *$T \cup \{v\}$ is an independent set of G,*

(ii) *for any two distinct vertices y, z in T, $N_C^+(z) \cap N(y) \cap V(C[y, z]) = \varnothing$ and $N(y) \cap N(z) \cap V(G-C) = \varnothing$.*

From Lemma 4, we can easily get that for any $w \in V(C_i)$ $(1 \leqslant i \leqslant t)$ and any two distinct vertices y, z in T,

(1) $|N(y) \cap V(C[w, x_{i+1}))| + |N(z) \cap V(C[w, x_{i+1}))| \leqslant |V(C[w, x_{i+1}))|$,

(2) $|N_C(y)| + |N_C(z)| \leqslant |C|$,

(3) $d_{G-C}(y) + d_{G-C}(z) \leqslant n - |C| - 1$.

If $|T| = \kappa$, that is $d(v) = \kappa$, then by (2) and (3) we obtain $d(y) + d(z) + d(v) \leqslant |C| + n - |C|\ 1 + \kappa = n + \kappa - 1$ for any two distinct vertices y, z of T, contrary to the fact that $\sigma_3 \geqslant n + \kappa$. Thus, we may assume that $|T| \geqslant \kappa + 1$.

Let V_0 be a vertex cut set of G with $|V_0| = \kappa$ and $V_1, ..., V_p$ be the vertex sets of components of $G - V_0$. Set $Y_i = V_i \cap T$ $(0 \leqslant i \leqslant p)$.

Since $|V_0| = \kappa < |T|$, we may assume that $Y_1 \neq \varnothing$. Select $u_i \in V_i$, with $u_i \in Y_i$ if $Y_i \neq \varnothing$ and, failing that, with $u_i \in V_i - (X \cup \{v\})$ if $V_i - (X \cup \{v\}) \neq \varnothing$ $(1 \leqslant i \leqslant p)$.

If $p \geqslant 3$, since $|T| \geqslant \kappa + 1$, by Lemma 4, we obtain

$$\sum_{i=1}^{3} d(u_i) \leqslant \sum_{i=1}^{3} |V_i| + 3\,|V_0| - \left|\left(\bigcup_{i=0}^{3} Y_i\right)\right| \leqslant n + 2\kappa - |T| \leqslant n + \kappa - 1,$$

which is contrary to the fact that $\sigma_3 \geqslant n + \kappa$.

If $p=2$, we consider two cases.

(a) $Y_2 \neq \emptyset$ or $Y_2 = \emptyset$ but $V_2 - (X \cup \{v\}) \neq \emptyset$.

By Lemma 4(i) and the choices of u_i $(i=1,2)$, we have

$$d(u_1)+d(u_2) \leqslant 2\,|V_0|+|V_1|+|V_2| - \left|\left(\bigcup_{i=0}^{2} Y_i\right) \cup \{v\}\right| \leqslant n+\kappa-|T|-1,$$

which implies $d(u_1)+d(u_2)+d(v) \leqslant n+\kappa-1$ as $d(v)=|T|$, a contradiction.

(b) $Y_2 = \emptyset$ and $V_2 \subseteq X \cup \{v\}$.

In this case, $v \notin V_2$, otherwise, $(X - V_2) \cup (X \cap V_2)^{-} \subseteq V_0$, implying that $|V_0| \geqslant |X| = \kappa+1$. Thus $v \in V_0$, and $V_2 \subseteq X = N(v)$, so $|V_0 - \{v\}| = \kappa - 1$ and $|Y_1| \geqslant 2$. Let $y, z \in Y_1$. By (1), for any $x \in V_2 \cap V(C_i)$ for some $1 \leqslant i \leqslant t$, we have $|N(y) \cap (V(C_i) - \{x\})| + |N(z) \cap (V(C_i) - \{x\})| \leqslant |C_i| - 1$. Thus $d(y)+d(z) \leqslant n-1-|V_2|$ by (1) and (3). On the other hand, $d(x) \leqslant |V_2| - 1 + \kappa$ for any $x \in V_2$. Hence, $d(x)+d(y)+d(z) \leqslant n+\kappa-2$, contrary to the fact that $\sigma_3 \geqslant n+\kappa$.

Therefore, the proof of Theorem 2 is complete.

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